Mixed strategies in games with ambiguity averse agents *

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Abstract

In normal form games, when agents exhibit ambiguity aversion the exclusion of mixed strategies from agents’ choice sets can enlarge the set of equilibria. While it is possible, in a game theoretic experiment, to enforce pure strategy reporting it is not possible to prevent subjects from mixing before reporting a pure strategy. This short paper establishes conditions under which the set of equilibrium in a game with ambiguity averse agents and pure strategy reporting is invariant to the existence of pre-play mixing devices. This result is crucial for the interpretation of recent experimental work on the role of ambiguity aversion in normal form games.

Keywords: Ambiguity Aversion, Mixed Strategies, Game Theory, Experimental Economics

JEL Codes: D81, C92, C72 and D03

1 Introduction

Consider a game between two agents that is mediated by a game theorist. The agents report their strategies to the game theorist, who then resolves the outcome of the game and pays the agents their winnings (or collects their losses). The game theorist may allow mixed strategy reports from the agents, and resolve the mixed strategy herself, or she may require that the agents report a pure strategy. If the game theorist requires pure strategy reports, as is typically the case in economic experiments, then the game theorist should be aware that the agents may still be using a mixed

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strategy that they are resolving privately before reporting. In the standard case, where agents have expected utility preferences, the set of Nash equilibrium under the two reporting requirements will be indistinguishable.

However, when agents have ambiguity averse preferences then the different reporting requirements may induce different games with different equilibria. Equilibrium concepts such as Lo (2009), Dow and Werlang (1994) and Eichberger and Kelsey (2000), which enforce pure strategy reporting, generate larger equilibrium sets than Lo (1996) which allows for mixed strategy reports. The difference arises because ambiguity averse preferences are non-linear and agents may have a strict preference for reporting mixed strategies. The equilibrium concepts that enforce pure strategy reporting assume that only pure strategies are available to agents: they either implicitly or explicitly rule out private pre-play mixing. By its very nature, however, private pre-play mixing will be unobservable to the game theorist and cannot readily be prevented.

How, then, does allowing for private pre-play mixing affect the pure strategy only equilibrium in papers such as Lo (2009)? The answer, provided in this paper, is that allowing for private pre-play mixing has no effect on the equilibrium set for agents with preferences that can be described by Choquet Expected Utility (Schmeidler, 1989) preferences with respect to a belief function. While this result is of independent interest, it is particularly relevant for experimental tests of ambiguity averse equilibrium concepts. Recent experiments have used equilibrium concepts that restrict agents to pure strategies, and thereby assume that their subjects are not engaging in pre-play mixing. Given that it is not possible to actively prevent subjects from pre-play mixing, the results in this paper are essential for a direct interpretation of the data in the previous experimental literature using pure strategy solution concepts.

To establish the main result, we reinterpret a formal mathematical result from Gilboa and Schmeidler (1994) (that a non-additive measure can be spanned by an appropriately chosen additive measure over a larger state space) and translate it from an individual decision making environment to a non-cooperative game. Each agent then plays a “mental” game that is closely related to, but distinct from, the game presented by the game theorist. The agent resolves their mixed strategy with respect to the mental game and reports the pure strategy realization to the game theorist, who then implements the strategy in the original game. The mental game is interpreted as a fictional accounting device used by the agent. Under this interpretation of a game, it is possible to rewrite a pure strategy equilibrium in a fashion that is consistent with pre-play mixing (see Section 4).

Interestingly, the main theorem of [Gilboa and Schmeidler (1994)] is a reinterpretation of an older result from cooperative game theory establishing that “the space of all non-additive measures (“games”) is spanned by a natural linear basis (“of unanamity games”). In this paper we explore how this cornerstone of cooperative game theory, when applied to decision making under uncertainty, can provide insights into the interpretation of mixed strategies in non-cooperative games where agents exhibit aversion to strategic uncertainty.

The rest of this paper is organized as follows. Section 2 introduces some mathematical tools from [Gilboa and Schmeidler (1994)] that are used in the rest of the paper. Section 3 introduces Lo-Nash equilibrium, and Section 4 presents an application of Lo-Nash equilibrium to the “mental” state space and establishes the key result of this paper. Section 5 concludes.

The use of the equilibrium from [Lo (2009)], and the resultant “mental” equilibrium, is illustrative. Other equilibrium concepts, which would generate distinct versions of the mental equilibrium defined here, could easily be used. The contribution, therefore, is to be found in the technique for modelling pre-play mixing of ambiguity averse agents rather than in our particular definition of mental equilibrium.

### 2 Preliminaries

Suppose that there exists a finite set of states of the world, $\omega \in \Omega$, and that an act, $f$, maps each state to an outcome in $\mathbb{R}$; that is $f : \Omega \rightarrow \mathbb{R}$. Choquet Expected Utility (CEU) generalizes Subjective Expected Utility by allowing a decision maker to hold non-additive beliefs which are represented by a capacity, $\nu$, defined over the set of events $\Sigma = 2^\Omega$. Suppose, without loss of generality, that for a given act, $f$, the set of states can be ordered so that $f(\omega_1) \geq f(\omega_2) \geq \ldots \geq f(\omega_n)$. A CEU agent calculates her utility of an act by evaluating the Choquet integral:

$$\int f \, d\nu = \int_0^\infty \nu(\{\omega | f(\omega) \geq t\}) \, dt + \int_{-\infty}^0 \left[ \nu(\{\omega | f(\omega) \geq t\}) - \nu(\Omega) \right] \, dt$$

We shall assume throughout that the capacities, $\nu$, are belief functions and we use $V$ to denote the space of all such capacities. That is, we assume that $\nu(\Sigma) = 1$ and that $\nu$ is totally monotone.

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2The equilibrium of [Eichberger and Kelsey (2000)] was also included in earlier versions of this paper. Not all ambiguity averse equilibrium concepts are appropriate however; the equilibrium of [Eichberger and Kelsey (2014)] is not compatible with our restriction that capacities must be belief functions.

3A capacity is totally monotone if $\nu(A) \geq 0$ for all $A \in \Sigma$ and, for every $A_1, A_2, \ldots, A_n \in \Sigma$, $\nu(\bigcup_{i=1}^n A_i) \geq \sum_{\{I \subseteq \{1, 2, \ldots, n\} \setminus \emptyset \}} (-1)^{|I|+1} \nu(\bigcap_{i \in I} A_i)$. Intuitively, a capacity is totally monotone if it is non-negative and the ca-
Under these assumptions it is also possible to represent the agent’s preferences using Maxmin Expected Utility (Gilboa and Schmeidler, 1994).

Throughout this paper, we shall rely on two basic mathematical results that are demonstrated in Gilboa and Schmeidler (1994). Firstly, the non-additive measure \( \nu \) can be spanned by an additive measure over an appropriately defined (larger) state space. Secondly, we can represent an agent with CEU preferences over acts that map from \( \Omega \) to \( \mathbb{R} \), as, equivalently, having SEU preferences over an appropriately transformed set of acts in the larger state space.

**Result 1** (Adapted from Gilboa and Schmeidler (1994)). For \( T, A \in \Sigma' = \Sigma \setminus \{\emptyset\} \), define

\[
e_T(A) = \begin{cases} 
1 & T \subseteq A \\
0 & \text{otherwise}
\end{cases}
\]

Then the set \( \{e_T\}_{T \in \Sigma'} \) forms a linear basis for \( V \). The unique coefficients \( \{\alpha_T^\nu\} \) satisfying

\[
\nu = \sum_{T \in \Sigma'} \alpha_T^\nu e_T
\]

are given by

\[
\alpha_T^\nu = \sum_{S \subseteq T} (-1)^{|T| - |S|} \nu(S)
\]

Furthermore, if \( \nu \) is totally monotone then \( \alpha_T^\nu \geq 0 \) for all \( T \in \Sigma' \) and if \( \nu \) is normalized then \( \sum_{T \in \Sigma'} \alpha_T^\nu = 1 \).

Result 1 provides the key building block for this paper: any non-additive measure over a state space can be spanned by an appropriately formed set of states constructed from the power set of the original state space. Furthermore, when the non-additive measure is a belief function then the spanning coefficients can be interpreted as probabilities over the newly constructed state space. Note the relationship between Result 1 and the proof of the representation theorem for CEU in Schmeidler (1989). In Result 1, we begin with a non-additive measure and ‘restore’ additivity by extending the state space. In Schmeidler (1989), the primitive is a SEU representation with respect to an additive measure, which is then extended to generate a CEU representation with respect to a non-additive measure. This tight relationship between Choquet Expected Utility and Subjective Expected Utility is formalized in the next result.

\[\text{Capacity of every event is larger than the sum of the capacities of all its sub-events (after accounting for the fact that each state is a member of multiple events).}\]
Recall that when $\nu$ is a belief function the core of $\nu$, denoted by $\text{Core}(\nu)$, is simply the set of probability measures, $p$, such that $p(A) \geq \nu(A)$ for all $A \in \Sigma$.

**Result 2** (Corollary 4.4 from Gilboa and Schmeidler (1994)). Suppose that $\nu$ is a belief function. Then for every $f \in F$

\[
\int f \, d\nu = \sum_{T \in \Sigma} \alpha^T \left[ \min_{\omega \in T} f(\omega) \right] = \min_{p \in \text{Core}(\nu)} \sum_{\omega \in \Omega} p(\{\omega\}) f(\omega). \tag{3}
\]

**Result 2** demonstrates that an agent with CEU preferences with respect to a belief function can have their preferences represented via either MEU or SEU preferences. While this relation between MEU and CEU preferences is both straightforward and well known, the representation with SEU preferences requires the formation of a new set of acts over the set $\Sigma = 2^\Omega$, with the outcome associated with each new act defined by the min function in Equation 3. We shall call these acts “mental” acts, and will sometimes refer to the event space as the “mental” state space. This terminology reflects that the event space may not be observable to an external observer and may, therefore, represent the mental accounting of the agent.

**Definition 1.** A mental act, $f'$, is an extension of an act, $f$, defined over the event space, $\Sigma' = 2^\Omega / \emptyset$, such that $f' : \Sigma' \mapsto \mathbb{R}$ with $f'(T) = \min_{\omega \in T} f(\omega)$ for all $T \in \Sigma'$.

It follows from **Result 2** that the preferences of an agent who has CEU preferences with respect to a belief function can be written in the expected utility form with respect to mental acts over the mental state space. The key feature of this preference representation is that preferences over mental acts are linear which, as discussed in Section A.2, implies that there is no preference for randomization over mental acts.

### 3 Lo-Nash Equilibrium

This section introduces Lo-Nash equilibrium, following Lo (2009) closely. Define a set of players $N = \{1, \ldots, n\}$, let each player $i \in N$ have a finite set of actions $A_i$, and define $A = \times_{i \in N} A_i$ and

\footnote{Note that, because each $p$ in the core is additive, the core can be equivalently defined with respect to $\Omega$ instead of $\Sigma$.}

\footnote{The interpretation of the event space as a mental “state” space implies that the agent, at least in his mental accounting, is not well calibrated about the nature of the world. In a sense, this is precisely the trade-off that allows us to move from non-linear preferences in the observable state space to linear preferences in the mental state space.}
\( A_{-i} = \times_{j \neq i \in N} A_j \). We shall endow each agent with a Von Neumann-Morgenstern utility function \( u : A \rightarrow \mathbb{R} \). Suppose that an agent has uncertainty regarding the strategy choices of their opponents, \( A_{-i} \). Then we can regard a strategy, \( a_i \), as an act over the state space \( A_{-i} \) generating a payoff \( u_i(a_i, a_{-i}) \) when the state \( a_{-i} \) is realized. As is standard, a game is defined by the tuple \((A_i, u_i)_{i \in N}\).

In a manner consistent with Gilboa and Schmeidler’s (1989) MEU formulation, we suppose that an agent’s beliefs regarding their opponents strategies are a closed and convex set of probability measures \( \Phi_i \subseteq \Delta(A_{-i}) \). Given \( \Phi_i \) an agent’s preferences are represented by

\[
\min_{\phi \in \Phi_i} \sum_{a_{-i} \in A_{-i}} u_i(a_i, a_{-i}) \phi_i(a_{-i}).
\]

Furthermore, we use \( \sigma \) to denote a probability measure on \( A \). We define \( \sigma^A_i(a_i) = \sum_{a_{-i} \in A_{-i}} \sigma(a_i, a_{-i}) \) as the marginal distribution of \( \sigma \) on \( A_i \) and \( \sigma^{A_{-i}}(a_{-i}) = \sum_{a_i \in A_i} \sigma(a_i, a_{-i}) \) as the marginal distribution of \( \sigma \) on \( A_{-i} \). Then, in the usual fashion we write \( \sigma(a_{-i}|a_i) = \frac{\sigma(a_i, a_{-i})}{\sigma^A_i(a_i)} \).

Finally, we write \( \text{supp} \sigma \) to denote the support of the probability distribution \( \sigma \), and define \( \text{supp} \Phi_i \) to be the union of the supports of the elements of \( \Phi_i \), and write \( \Phi \) to denote the profile \((\Phi_i)_{i \in N}\). We are now ready to define a Lo-Nash equilibrium.

**Definition 2** (Lo-Nash equilibrium). A pair \(<\sigma, \Phi>\) forms a Lo-Nash equilibrium if it satisfies

\[
\sigma(\cdot|a_i) \in \Phi_i \quad \forall a_i \in \text{supp} \sigma^A_i, \forall i \in N \tag{5}
\]

\[
\text{supp} \Phi_i = \times_{j \neq i} \text{supp} \sigma^{A_j} \quad \forall i \in N \tag{6}
\]

and

\[
a_i \in \arg \max_{\tilde{a}_i \in A_i} \min_{\phi_i \in \Phi_i} \sum_{a_{-i} \in A_{-i}} u_i(\tilde{a}_i, a_{-i}) \phi_i(a_{-i}) \quad \forall a_i \in \text{supp} \sigma^A_i, \forall i \in N \tag{7}
\]

Equation 7 requires that all strategies that are played in an equilibrium are best responses among the set of pure strategies, with preferences defined as MEU preferences with respect to the equilibrium conjectures \( \Phi \). Equations 5 and 6 are the consistency requirements: Equation 6 ensures that a strategy is played with a positive probability iff it is expected to be played with a positive probability, and Equation 5 forces actual strategies to be contained in the belief sets. Note that Equation 5 allows for conditioning of \( \sigma \) on \( a_i \) - this allows for strategies to be correlated, but the realized strategy must lie within player \( i \)'s belief set for all \( a_i \). Nash equilibrium is a special case of Lo-Nash equilibrium, thereby ensuring existence of Lo-Nash equilibrium for all finite normal form games.
Lo explicitly restricts attention to pure strategies to prevent agents from holding a strict preference for mixing. Mixed equilibrium are, therefore, interpreted as equilibrium in beliefs. In the following section we demonstrate that by reformulating the problem using the mental state space, we can allow for explicit pre-play mixing without altering the equilibrium set.

4 Mental games and an application of Lo-Nash equilibrium

We wish to extend the Lo-Nash environment to allow for (pre-play) mixed strategies without affecting the equilibrium set. The standard implementation of mixed strategies as a state-by-state linear combination over the state space \( \Omega_i = A_{-i} \) is not appropriate as it will induce a strict preference for mixed strategies that fundamentally alters the strategic forces in the game. Instead, we return to the leading example of an agent who must provide a pure strategy report to the game theorist, but may resolve a mixed strategy prior to reporting their pure strategy. We proceed by postulating that each agent faces a “mental” problem of deciding which (potentially) mixed strategy to play, and that the relevant state space is \( \Sigma_i' = 2^{A_{-i}} \setminus \{\emptyset\} \), with their extended utility function defined, for all \( T \in \Sigma_i' \), by:

\[
u'(a_i, T) = \min_{a_{-i} \in T} u(a_i, a_{-i}).\]

Notice that when \( T \in A_{-i} \) then \( u'(a_i, T) = u(a_i, a_{-i}) \) so that the extended utility function is consistent with the original utility function.

A justification for the use of the mental state space is warranted. The use of a mental state space allows an agent’s beliefs over their opponents actions to be divorced from the confidence the agent has in those beliefs, thereby allowing the agent to express strategic uncertainty in a natural fashion. As an example, consider an opponent that has three strategies \( \{A, B, C\} \) where \( C \) is dominated. The agent knows that the opponent will not play \( C \), but is uncertain about the mixing probability that the opponent will use across \( A \) and \( B \). In a mental game these beliefs can be represented by placing a weight of one on the state \( T = \{A, B\} \) and a weight of zero on all other states, while the standard formulation does not allow an agent to express strategic uncertainty in this fashion. A standard SEU agent can still be accommodated – they simply place zero weight on all non-singleton events – so that the mental problem generalizes the problem facing an agent in a standard game.

A game in the mental state space is defined by the tuple \( (A_i, u'_i)_{i \in N} \). We also define the natural
extension of $\sigma$ over $\Sigma_i$: $\sigma(a_i, T) = \sum_{a_{-i} \in T} \sigma(a_i, a_{-i})$ for all $T \in \Sigma_i$. Finally, we introduce the probability measure $\alpha_i \in \Delta(\Sigma_i)$ which can be interpreted as agent $i$'s belief over the event space $\Sigma'_i$. We write $\alpha$ to denote the profile $(\alpha_i)_{i \in N}$.

We are now ready to define our mental equilibrium. In essence mental equilibrium is simply an application of Result 1 to Lo-Nash equilibrium, as we establish in Theorem 1 below.

**Definition 3 (Mental equilibrium).** A pair $<\sigma, \alpha>$ form a mental equilibrium if

\[
\sigma(T|a_i) \geq \sum_{\tau \in T} \alpha_i(\tau) \quad \forall T \in \Sigma_i, \forall a_i \in \text{supp } \sigma^{A_i}, \forall i \in N \tag{8}
\]

\[
\sum_{\{T: a_{-i} \notin T\}} \alpha_i(T) = 1 \Leftrightarrow \prod_{j \neq i} \sigma^{A_j}(a_j) = 0 \quad \forall a_{-i} \in A_{-i}, \forall i \in N \tag{9}
\]

and

\[
a_i \in \text{arg max } \sum_{T \in \Sigma_i} u'_i(\hat{a}_i, T) \alpha_i(T) \quad \forall a_i \in \text{supp } \sigma^{A_i}, \forall i \in N \tag{10}
\]

Note that the existence of mental equilibrium is guaranteed whenever a Nash equilibrium exists. Furthermore, the formulation of the mental equilibrium is linear in all parameters, which implies that solving for equilibrium is a linear complementarity problem – a class of problems that have been well studied. In particular, if the game theorist is willing to assume a support for $\sigma$, then solving for equilibrium reduces to solving a linear program.

While the interpretation of mixed strategy Lo-Nash equilibrium must necessarily be as an equilibrium in beliefs, that is not the case for a mental equilibrium. The reason is that the preferences expressed in Equation 10 are linear, while the preferences expressed in Equation 7 are not. To be explicit $\sum_{T \in \Sigma_i} u'_i(\beta a_i + (1 - \beta) a'_i, T) \alpha_i(T) = \beta \sum_{T \in \Sigma_i} u'_i(a_i, T) \alpha_i(T) + (1 - \beta) \sum_{T \in \Sigma_i} u'_i(a'_i, T) \alpha_i(T)$, which ensures that whenever an agent is indifferent between two pure strategies they are also indifferent between any mix between the two strategies. It is crucial, however, that the randomization is resolved prior to the game being played, precisely because the formulation of the mental state space explicitly relies on the agent facing strategic uncertainty. If the randomization is realized after strategies have been declared, then at the moment of realization there is no longer any strategic uncertainty and the use of the mental state space is problematic.

Mental equilibrium, therefore, is an equilibrium concept that accommodates pre-play mixing.

We now turn to our main result, linking Lo-Nash equilibrium and mental equilibrium.

\[\text{[Footnote: Similar arguments regarding the timing of the realization of uncertainty for individual decision making problems have recently been made in Seo (2009), Azrieli et al. (2016), Baillon et al. (2014) and Eichberger et al. (2016), among others.]}\]
**Theorem 1.** If \( <\sigma,\alpha> \) is a mental equilibrium then there exists a \( \Phi \) such that \( <\sigma,\Phi> \) is a Lo-Nash equilibrium. Conversely, if \( <\sigma,\Phi> \) is a Lo-Nash equilibrium and \( \Phi_i \) is the core of a belief function for all \( i \) then there exists an \( \alpha \) such that \( <\sigma,\alpha> \) is a mental equilibrium.

The restriction in the second of part of Theorem 1, that \( \Phi_i \) is the core of a belief function, is best interpreted as a restriction on preferences rather than beliefs. As Result 2 establishes, this is a restriction to the case where agents preferences can be equivalently represented by any of MEU, CEU or the linear representation where all \( \alpha > 0 \). In practice, the restriction is often non-binding – the game studied in Calford (2016) is an example. Given this restriction on preferences, every Lo-Nash equilibrium is a mental equilibrium and vice versa. Thus, we have established the robustness of experiments on ambiguity aversion in games to the existence of pre-play mixing – any Lo-Nash equilibrium, under an assumption that only pure strategies are available, remains an equilibrium when agents have access to pre-play randomization devices.

### 5 Conclusions

This short paper presents a methodology for extending pure strategy only ambiguity averse equilibrium concepts to allow for (pre-play) mixed strategies.

The theoretical interest in the structure and interpretation of mixed strategy equilibrium for agents with uncertainty averse preferences is readily apparent. Unlike SEU agents, ambiguity averse agents are not necessarily indifferent between a mixed strategy and the pure strategy supports of the mixed strategy. Indeed, a majority of ambiguity averse equilibrium concepts explicitly restrict their analysis to pure strategies. In this context, the methodology introduced here can be viewed as an equilibrium-preserving interpretation of mixed strategies: when agents are ambiguity averse and have access to a mixing device that resolves before the strategic interaction occurs then, under a particular assumption on preferences, the set of equilibrium in games with and without pre-play mixing are identical. This result can be used to provide further justification of the results presented in recent papers that study ambiguity aversions in games using laboratory experiments.
A Appendix

A.1 Proof of Theorem 1

Lemma 1. Suppose that $\nu$ is a belief function and $\Phi = \text{core}(\nu)$. Then $A \in \text{supp} \Phi$ if and only if $\nu(A^c) < 1$.

Proof of Lemma 1. Suppose that $A \in \text{supp} \Phi$. Therefore there exists a $\phi \in \Phi$ such that $\phi(A) > 0$ and $\phi(A^c) < 1$. Therefore there exists a $p \in \text{core}(\nu)$ such that $p(A^c) < 1$. Therefore $\nu(A^c) < 1$.

Suppose that $A /\in \text{supp} \Phi$. Therefore $\phi(A) = 0$ for all $\phi \in \Phi$. Therefore $p(A^c) = 1$ for all $p \in \text{core}(\nu)$. Note that there always exists a $p \in \text{core}(\nu)$ such that $p(A^c) = \nu(A^c)$. Therefore $\nu(A^c) = 1$.

Lemma 2 (Lo-Nash equilibrium). Consider a pair $< \sigma, \Phi >$ such that $\Phi_i = \text{core} \nu_i$ for all $i$, where $\nu_i$ is a belief function. If $\sigma$ and $\nu_i$ satisfy:

$$\sigma|_i \in \text{core} \nu_i \quad \forall a_i \in \text{supp} \sigma A_i, \forall i \in N$$

$$\nu_i(a^c_i) = 1 \iff \prod_{j \neq i} \sigma A_j(a_j) = 0 \quad \forall a_{-i} \in A_{-i}, \forall i \in N$$

and

$$a_i \in \arg \max \int \hat{a}_i d\nu_i \quad \forall a_i \in \text{supp} \sigma A_i, \forall i \in N$$

then the pair $< \sigma, \Phi >$ form a Lo-Nash equilibrium. Conversely, if $< \sigma, \Phi >$ are a Lo-Nash equilibrium then $\sigma$ and $\nu_i$ satisfy Equations [A.1], [A.2] and [A.3]

Proof of Lemma 2. We need to demonstrate that $< \sigma, \Phi >$ is a Lo-Nash equilibrium.

Equation A.1 is equivalent to Equation 5.

The equivalence of Equation 7 and equation [Equation A.3] follows directly from Result 2.

$$\prod_{j \neq i} \sigma A_j(a_j) = 0 \iff \exists j \neq i \text{ s.t. } \sigma A_j(a_j) = 0 \iff a_{-i} \notin \times_{j \neq i} \text{supp } \sigma A_j$$

and Lemma 1 establish that equation 6 and equation A.2 are contrapositives.

Proof of Theorem 1. Given the above Lemmas, it is sufficient to show an equivalence between Equations A.1 and 8, Equations A.2 and 9, and Equations A.3 and 10. We begin by noting that
Result 1 ensures that each $\alpha_i$ can be associated with a belief function $\nu_i$ (and vice-versa) and that $\Phi_i$ can be defined as the core of $\nu_i$.

The equivalence of Equations \ref{conversionA} and \ref{conversion} follows from Results 1 and 2.

The equivalence of Equations \ref{conversionA} and \ref{conv} is a consequence of the definition of the core of a capacity and the fact that $\nu_i(B) = \sum_{T \subseteq B} \alpha_i(T)$ (which follows directly from the definition of $\alpha_i$).

The equivalence of Equations \ref{conversionA} and \ref{conv} follows immediately from $\nu_i(a_{-i}^c) = \sum_{T: a_{-i} \notin T} \alpha_i(T)$ which again is an immediate consequence of the definition of $\alpha_i$.

\begin{proof}
\end{proof}

\section{A.2 Example of mixing as a hedge against uncertainty}

To illustrate the role of mixing in the mental state space we present a simple example. Consider a world with two states $\Omega = \{U, D\}$ and two acts $l$ and $r$. The agent earns a payoff of 1 if they choose $l$ when the state is $U$ or choose $r$ when the state is $D$ and nothing otherwise\footnote{Although this example is being presented solely as an individual decision maker problem, it should be clear that the setup could also describe a standard coordination game as viewed by a single player (the opponent’s payoffs have been suppressed, but they are not material to the current discussion).}. Further, suppose that the agent has complete subjective uncertainty regarding the true state of the world, so that $\nu(U) = \nu(D) = 0$ and $\nu(\{U, D\}) = 1$. The agent then holds preferences such that $l \sim r \sim 0$ (where 0 should be understood as an act that pays 0 in every state).

In the Lo (1996) framework, which defines mixtures state-by-state, the mixture $\alpha r + (1-\alpha) l$ will earn a payoff of $\alpha$ in state $D$ and $1 - \alpha$ in state $U$. Therefore, mixing can provide a hedge against uncertainty: the strategy $\frac{1}{2} r + \frac{1}{2} l$ earns a payoff of $\frac{1}{2}$ in each state so that $\frac{1}{2} r + \frac{1}{2} l \sim \frac{1}{2} \sim l \sim r$.

Now, consider the mental state space constructed from $\Omega$: $\Sigma = \{U, D, \Omega\}$. An application of Result 1 generates $\alpha_U^\nu = \alpha_D^\nu = 0$ and $\alpha_\Omega^\nu = 1$. The act $l'$ pays 1 in state $U$ and nothing in states $D$ and $\Omega$, and the act $r'$ pays 1 in state $D$ and nothing in states $U$ and $\Omega$. In this environment, state-by-state mixing does not provide a hedge against uncertainty: the act $\frac{1}{2} r' + \frac{1}{2} l'$ earns $\frac{1}{2}$ in states $U$ and $D$, but only earns 0 in state $\Omega$. Given that $\alpha_\Omega^\nu = 1$ then $\frac{1}{2} r' + \frac{1}{2} l' \sim l' \sim r' \sim 0$. The existence of the additional state, $\Omega$, in the mental state space implies that state-by-state mixing cannot provide a hedge against the uncertainty.

The result is general. To see this, consider arbitrary acts, $f'$ and $g'$, in the mental state space. Denote an arbitrary mix between $f'$ and $g'$ as $\beta f' + (1 - \beta) g'$. Then $U(\beta f' + (1 - \beta) g') = \frac{1}{2} r' + \frac{1}{2} l' \sim l' \sim r' \sim 0$. The
\[
\sum_{T \in \Sigma'} \alpha^\nu_T (\beta f' + (1-\beta)g') = \sum_{T \in \Sigma'} (\alpha^\nu_T \beta f' + \alpha^\nu_T (1-\beta)g') = \beta \sum_{T \in \Sigma'} \alpha^\nu_T f' + (1-\beta) \sum_{T \in \Sigma'} \alpha^\nu_T g' = \beta U(f') + (1-\beta)U(g').
\]

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